**RANDOM PROCESSES**



What is a ‘***process’*** as opposed to a ‘variable’?

Where do we most commonly encounter the word ‘process'?

**Definition**: A *random process* is a collection (a.k.a. *ensemble*) of random variables which are indexed by time t. A random process is denoted as { X(t) }, where, for each value of t, X(t) or Xt is an RV with sample space S.

The set of all time instants T is called the *index set*, or *parameter set*.

Both S and T can be *discrete* or *continuous*, independently of each other. But we shall consider only random processes in which both are discrete. Such random processes are known as ***discrete random sequences***.

To understand the concept better, one should imagine a trial or experiment being carried out repeatedly at each time instant t.

Simple example:

Rolling an unbiased die repeatedly, at every discrete time instant t.

Then S = { 1, 2, 3, 4, 5, 6 }.

Let Xn represent the outcome of the nth roll. Then { Xn | n > 1 } is a discrete random sequence.



\*\*\*

A given such sequence which is ‘realized’ in practice is known as a *realization*.

Note that we have not said anything yet about the probability distribution of the RVs Xn. We have NOT assumed that Xn and Xm, for m ~= n, have the same probability distribution, or that they are independent of each other, etc.

The Xn may or may not have identical probability distributions, and they may or may not be independent of each other. In the simple example above, however, both these properties hold. We shall consider mainly such **i.i.d**. examples – that is, ***independent and identically distributed*** RVs.

**Probability relationships & classes of processes**

For a discrete random sequence, F(x,t) = Prob[ X(t) < x ] is known as the *first order distribution* of the process { X(t) }.

Note that F(x,t) is nothing but the cumulative distribution function of the random variable X(t). We also assume that the { X(t) } are i.i.d.

If the distributions of { X(t) } are independent of time t, the random process is said to be ***stationary***.

Markov process

Let t1 < t2 < t3 ... < tn < tn+1 = t. If

Prob[ X(t) < x | X(t1) = x1 & X(t2) = x 2 & ... & X(tn) = xn ]

=

Prob[ X(t) < x | **X(tn) = xn** ]

Then the random process { X(t) } is known as a (first order) Markov process.

In simple language, in a Markov process, the behaviour of the process at time tn+1 depends only on its value at time tn; *not* on earlier values.

In this case, the value of X(tn) or Xn is known as the *state* of the process at time instant tn.

In a Markov process, the future evolution of the process beyond tn depends only on its state at tn. Specifically, Xn+1 depends only on Xn, not earlier states.

Since we have assumed time t to be discrete, a discrete Markov process is also known as a Markov chain.

We must assume that the distribution of { X(t) } has been learnt from analysis of process realization(s).

**Mean of a random process**

m(t) = Exp[ X(t) ]

Of course, for a stationary process, m(t) = m, for all t.

Similarly for variance, as we have seen for random variables.

We are considering only i.i.d. processes, and further we also assume the following property:

any statistical property of a specific X(t), for a time instant t

=

that same property as calculated from a realization of the process

This property, known as ***ergodicity***, actually also follows from our earlier assumption that the distribution of { X(t) } has been learnt from analysis of process realization(s).

[NOTE: In general, in nature, it is not necessary that a random process must be i.i.d., stationary and/or ergodic. Various combinations can possibly occur; such possibilities / impossibilities define a different subject matter.]

**Examples**

1. Rolling a pair of dice. X(t) = number rolled at time t.

2. Rolling a pair of normal dice: one **green** and one **red**, with X(t) = G(t) - R(t). The sample space now is S = { -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5 }.

Note that, in these two and other similar examples, P[ X(t) < x ] is independent of the prior values of X(). ‘Zeroth order' Markov process.

3. ‘Random walk’ process is a first order Markov process – to be discussed later. May serve as a simple model for stock prices.